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On the excitation of edge waves on beaches

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The excitation of standing edge waves of frequency $\frac{1}{2}\omega$ by a normally incident wave train of frequency ω has been discussed previously (Guza & Davis 1974; Guza & Inman 1975; Guza & Bowen 1976) on the basis of shallow-water theory. Here the problem is formulated in the full water-wave theory without making the shallow-water approximation and solved for beach angles $\beta = \pi/2N$, where N is an integer. The work confirms the shallow-water results in the limit $N \gg 1$, shows the effect of larger beach angles and allows a more complete discussion of some aspects of the problem.

1. Introduction

Guza & Davis (1974) discuss how standing edge waves may be formed on beaches through the instability of incident wave trains. They show, using the shallow-water approximation, that if a normally incident wave train of frequency ω is perturbed by a small disturbance in the form of an edge wave of frequency $\frac{1}{2}\omega$, nonlinear coupling produces energy transfer and a growth of the edge-wave mode. The small perturbation theory predicts exponential growth, but when the amplitudes become comparable further nonlinear interaction terms become important and limit the final amplitudes. Recently Guza & Bowen (1976), again from the shallow-water approximation, have estimated the ultimate steady-state amplitudes and found good agreement with the experimental observations of Guza & Inman (1975).

We studied the instability problem using the full water-wave theory, without making the shallow-water approximation. The original interest was in questions of the non-uniformity of the shallow-water approximation in the deep water away from the shore, since similar questions arose in earlier work on travelling edge waves (Whitham 1976; Minzoni 1976). In the present case the non-uniformities turn out to be mild, are easily corrected and do not affect the main results for small beach angles. However, more interestingly, we were able to include all the relevant interaction terms and trace the whole edge-wave development from initial instability to final steady state. Although the main results for small beach angles have in the meantime been covered by the shallow-water theory of Guza & Bowen, it is worthwhile to present an account of the full theory. It endorses the shallow-water estimates, shows the effect of larger beach angles, and relates the edge-wave amplitudes to given incoming waves at infinity (which can not be done directly in the shallow-water theory).

Guza & Bowen (1975) discuss the instability of incoming wave trains at oblique incidence. The methods developed here might also be extended to this case.

2. Formulation of the interaction equations

In terms of a velocity potential $g\Phi/\omega$, the problem is to solve

$$\Phi_{xx}+\Phi_{yy}+\Phi_{zz}=0 \tag{1}$$

in the wedge $-y\tan\beta<z<0$, subject to the boundary conditions

$$\Phi_y\sin\beta+\Phi_z\cos\beta=0, \quad z=-y\tan\beta, \tag{2}$$

$$\Phi_z+\frac{1}{g}\Phi_{tt}=-\frac{1}{\omega}\{(\nabla\Phi)^2\}_t+\frac{1}{\omega}\left\{\left(\Phi_z+\frac{1}{g}\Phi_{tt}\right)\Phi_t\right\}_z+C\{\Phi\}, \quad z=0, \tag{3}$$

where x, y and z are longshore, offshore and vertical co-ordinates, respectively, and β is the beach angle. The nonlinear boundary conditions on the surface have been expanded in powers of Φ and its derivatives, and transformed into equivalent conditions on the mean surface $z=0$; $C\{\Phi\}$ indicates third-order terms, which will be included in detail when needed. The surface elevation is given by

$$\zeta=-\frac{1}{\omega}\Phi_t+\left\{\frac{1}{2\omega^2}(\Phi_t^2)_z-\frac{g}{2\omega^2}(\nabla\Phi)^2\right\}+\dots. \tag{4}$$

We describe the incoming wave and its reflexion by a potential $\frac{1}{2}\phi(y,z,t)\,e^{i\omega t}$ + complex conjugate, and consider its interaction with an edge wave described by $\frac{1}{2}\chi(x,y,z,t)\,e^{\frac{1}{2}i\omega t}$ + c.c. The dependence of ϕ and χ on t will arise from slow variations due to the nonlinear interactions. The potentials ϕ and χ must separately satisfy Laplace's equation in the wedge and the bottom boundary condition (2). The interaction equations are provided by the nonlinear condition (3) on $z=0$ and may be written to the appropriate approximation in the form

$$\chi_z-\frac{\omega^2}{4g}\chi+\frac{i\omega}{g}\chi_t=(\phi,\chi^*)+(\chi,\chi,\chi^*), \tag{5}$$

$$\phi_z-(\omega^2/g)\,\phi=(\chi,\chi). \tag{6}$$

The quadratic interaction terms indicated by (ϕ,χ^*) arise because products of $\phi e^{i\omega t}$ with the conjugate $\chi^* e^{-\frac{1}{2}i\omega t}$ produce terms in $e^{\frac{1}{2}i\omega t}$, and so contribute to changes in $\chi e^{\frac{1}{2}i\omega t}$. Similarly, products of $\chi e^{\frac{1}{2}i\omega t}$ with itself contribute to $\phi e^{i\omega t}$ and appear on the right of (6). The derivative χ_t will be required in (5) and represents the growth of the edge wave. However, the growth rate is small and the further term χ_{tt}, which also arises from the left-hand side of (3), is of smaller order and is neglected. The t derivatives of ϕ are all of smaller order and are neglected in (6); however, ϕ still varies with time, reacting to the growth of χ. When the amplitude of χ becomes relatively large, cubic products of $\chi e^{\frac{1}{2}i\omega t}$, $\chi e^{\frac{1}{2}i\omega t}$ and $\chi^* e^{-\frac{1}{2}i\omega t}$ which also contribute to the subharmonic $e^{\frac{1}{2}i\omega t}$ must be included on the right of (5).

The consistency of the orders of approximation will appear in the further discussion.

3. The instability problem

For the discussion of the initial instability, χ is taken to be small compared with ϕ. The feedback of χ on ϕ described by (6) is neglected, and in (5) we take ϕ to be the undisturbed wave train $\phi_0(y,z)$. The cubic interaction term is also neglected, so (5) is approximated as

$$\chi_z - \frac{\omega^2}{4g}\chi = -\frac{i\omega}{g}\chi_t + (\phi_0, \chi^*). \tag{7}$$

The linear edge wave

$$\chi_1 = bE(y,z)\cos kx, \quad E = \exp(-ky\cos\beta + kz\sin\beta), \tag{8}$$

with

$$k\sin\beta = \omega^2/4g, \tag{9}$$

satisfies (1), (2) and

$$\chi_{1z} - (\omega^2/4g)\chi_1 = 0, \quad z = 0. \tag{10}$$

In a successive approximation

$$\chi = \{b(t)E(y,z) + \chi_2 + \ldots\}\cos kx, \tag{11}$$

the correction term χ_2 satisfies

$$\chi_{2z} - k\sin\beta\,\chi_2 = -\frac{i\omega}{g}E\frac{db}{dt} + (\phi_0, E)\,b^* \tag{12}$$

on $z = 0$. Since E is a solution of the homogeneous problem, there is a bounded solution for χ_2 if and only if the right-hand side R of (12) satisfies the appropriate orthogonality condition

$$\int_0^\infty [ER]_{z=0}\,dy = 0. \tag{13}$$

This is similar to the well-known Fredholm alternative theorem for the simpler eigenvalue problems of ordinary differential equations (Courant & Hilbert 1953, p. 359). A derivation of the necessity of (13) is given in the appendix. For our cases, the sufficiency can be shown by the explicit construction of a bounded solution for χ_2; the full details are not given, although one important aspect of the solution is discussed in § 9.

If the term db/dt is omitted in (12), the orthogonality condition can not be satisfied; the forcing term (ϕ_0, E) 'resonates' with the basic mode. This accounts for the inclusion of the time dependence and the growth of the edge wave mode. The orthogonality condition then provides an equation to determine $b(t)$, namely†

$$\frac{i\omega}{g}\frac{db}{dt}\int_0^\infty E^2\,dy = b^*\int_0^\infty (\phi_0, E)\,E\,dy. \tag{14}$$

This shows exponential growth on a time scale a_0^{-1}, where a_0 is the amplitude of ϕ_0.

We return to the evaluation of the integral in (14) and discuss the dependence of the growth rate on β after deriving the full interaction equations.

† The integrals in (14) and similar ones below are to be evaluated on $z = 0$. To simplify the notation, this will not always be indicated explicitly.

4. The full interaction equations

As the amplitude of χ increases, the nonlinear coupling of (χ, χ) with ϕ in (6) eventually becomes important. This stage is reached when ϕ and (χ, χ) are of the same order; since $\phi = O(a_0)$, χ is then $O(a_0^{\frac{1}{2}})$. The cubic term (χ, χ, χ^*) in (5) must also be included, since it is of the same order as (ϕ, χ^*). With these coupling terms, there is the possibility that a final steady state is attained with $\chi = O(a_0^{\frac{1}{2}})$.

When the various orders of magnitude are incorporated, we may take a more formal expansion

$$\chi = \{a_0^{\frac{1}{2}}\chi^{(1)}(y,z,T) + a_0^{\frac{3}{2}}\chi^{(2)}(y,z,T) + \ldots\}\cos kx, \tag{15}$$

$$\phi = a_0\phi^{(1)}(y,z,T) + a_0^2\phi^{(2)}(y,z,T) + \ldots, \tag{16}$$

where $T = a_0 t$. When these are substituted in (5) and (6), with $\omega^2/4g = k\sin\beta$, we have

$$\chi_z^{(1)} - k\sin\beta\chi^{(1)} = 0, \tag{17}$$

$$\chi_z^{(2)} - k\sin\beta\chi^{(2)} = (-i\omega/g)\,\chi_T^{(1)} + (\phi^{(1)}, \chi^{(1)*}) + (\chi^{(1)}, \chi^{(1)}, \chi^{(1)*}), \tag{18}$$

$$\phi_z^{(1)} - (\omega^2/g)\,\phi^{(1)} = (\chi^{(1)}, \chi^{(1)}). \tag{19}$$

At this stage the consistency of the terms kept in (5) and (6) is verified by the formal expansions (15) and (16).

The solution of (17) is taken to be the edge wave

$$\chi^{(1)} = B(T)\,E(y,z), \tag{20}$$

where E is the exponential given in (8). After (20) has been substituted in (18) and the expressions for the quadratic and cubic interaction terms introduced explicitly (see Whitham 1976, equation (A 3), for the cubic terms), we have

$$\chi_z^{(2)} - k\sin\beta\chi^{(2)} = -(i\omega/g)\,EB_T - i\{\tfrac{1}{2}\phi_y^{(1)}E_y + \tfrac{1}{2}\phi_z^{(1)}E_z + \tfrac{1}{4}[(\phi_z^{(1)} - 4k\sin\beta\phi^{(1)})\,E]_z\}\,B^*$$
$$-\frac{3}{16}\frac{\cos 2\beta}{\sin\beta}k^3E^3B^2B^*. \tag{21}$$

From (19) and (20),

$$\phi_z^{(1)} - (\omega^2/g)\,\phi^{(1)} = -\tfrac{1}{2}ik^2E^2B^2. \tag{22}$$

The orthogonality condition (13) is now applied to the right-hand side of (21). After some manipulation (given in the appendix), the condition reduces to

$$\frac{dB}{dT} = \frac{\omega^3\cos\beta}{4g\sin^2\beta}\left\{\left(k\int_0^\infty \phi^{(1)}E^2\,dy\right)B^* - \frac{5ik}{64\sin 2\beta}B^2B^*\right\}. \tag{23}$$

In discussing the solutions, we revert to the unnormalized variables $b(t) = a_0^{\frac{1}{2}}B(T)$, $t = T/a_0$ and $\phi = a_0\phi^{(1)}$. The amplitude parameter a_0 factors out, of course, because the terms are of the same order in a_0, and we have

$$\frac{db}{dt} = \frac{\omega^3\cos\beta}{4g\sin^2\beta}\left\{\left(k\int_0^\infty \phi E^2\,dy\right)b^* - \frac{5ik}{64\sin 2\beta}b^2b^*\right\}. \tag{24}$$

This is the equation for $b(t)$, where ϕ must be found from the boundary-value problem

$$\phi_{yy} + \phi_{zz} = 0, \quad -y\tan\beta < z < 0, \tag{25}$$

$$\phi_y \sin\beta + \phi_z \cos\beta = 0, \quad -y\tan\beta = z, \tag{26}$$

$$\phi_z - (\omega^2/g)\,\phi = -\tfrac{1}{2}ik^2E^2b^2, \quad z = 0. \tag{27}$$

The experimental work of Guza & Inman (1975) and others indicates edge-wave generation when the incoming wave is strongly reflected without much breaking. This is fortunate for the theory, since the solutions for ϕ can be taken to be regular at the shoreline, and the rather imprecise use of singular solutions to represent breaking is not required for the situation of most interest.

It is natural in this problem to specify the amplitude a_∞ of the incoming wave train at infinity. However, since we now consider only regular solutions for ϕ, an equivalent parameter is the amplitude of the original undisturbed solution ϕ_0 at the shoreline. This will be taken as the a_0 of the above discussion, since it is the relevant parameter for comparing the various interaction terms. From (4), a_0 is also the amplitude of the surface elevation ζ to first order. In the full theory, a_0 and a_∞ are related by a factor depending on β. With this choice of a_0, the undisturbed potential is written as

$$\phi_0 = a_0 S_0(y, z), \tag{28}$$

where $S_0(y,z)$ is a solution of the homogeneous problem normalized to have $S_0(0,0) = 1$.

5. Solution of the instability problem

For the instability problem, b is taken to be small compared with $a_0^{\frac{1}{2}}$ and the cubic term b^2b^* is neglected in (24). The coupling term on the right of (27) is also neglected and ϕ is taken to be the undisturbed incident wave ϕ_0. Then we have

$$db/dt = \tilde{\gamma} b^*, \tag{29}$$

where
$$\tilde{\gamma} = a_0 \frac{\omega^3 \cos\beta}{4g\sin^2\beta}\gamma(\beta), \quad \gamma(\beta) = k\int_0^\infty S_0(y,0)\exp(-2ky\cos\beta)\,dy. \tag{30}$$

From (29),
$$d^2b/dt^2 = \tilde{\gamma}\, db^*/dt = \tilde{\gamma}\tilde{\gamma}^* b;$$

the complex amplitude b grows exponentially at a rate $|\tilde{\gamma}|$ proportional to a_0.

The function $S_0(y,z)$ is one of the eigensolutions $S_l(y,z)$, $0 < l < \infty$, of the boundary-value problem

$$S_{yy} + S_{zz} = 0, \quad -y\tan\beta < z < 0, \tag{31}$$

$$S_y \sin\beta + S_z\cos\beta = 0, \quad -y\tan\beta = z, \tag{32}$$

$$S_z - lS = 0, \quad z = 0. \tag{33}$$

At present only the solution for $l = l_0 = \omega^2/g$ is required, but we use the extended notation in discussing the solution since the inhomogeneous problem for ϕ will be solved later by an integral over the continuous spectrum $0 < l < \infty$. The normalization is chosen to be $S_l(0,0) = 1$ in agreement with (28).

For arbitrary β, $S_l(y,z)$ is known in terms of contour integrals (see Stoker 1957, chap. 5), but we find more explicit results by taking the special values $\beta = \pi/2N$,

N = integer, where the solution is in fact the sum of exponentials (Hanson 1926). These values of β are sufficient for our purposes, since the variation of $\gamma(\beta)$ and other integrals required later is not very great.

For $\beta = \pi/2N$, a form of the solution given by Friedrichs (1948) is particularly useful here. It is

$$S_l(y,z) = \frac{1}{4\pi i}\int_{\mathscr{C}} \frac{\xi^{N-1}\, e^{\xi l(y+iz)}\, d\xi}{(\xi-\xi_1)\dots(\xi-\xi_N)} + \frac{1}{4\pi i}\int_{\mathscr{C}} \frac{\xi^{N-1}\, e^{\xi l(y-iz)}\, d\xi}{(\xi-\xi_0)\dots(\xi-\xi_{N-1})}, \tag{34}$$

where
$$\xi_n = i\, e^{n\pi i/N}, \quad n = 0, \dots, N, \tag{35}$$

and $\mathscr{C}$ may be taken as a closed contour around the poles. On the surface $z = 0$, the integrals combine into

$$S_l(y,0) = \frac{1}{2\pi i}\int_{\mathscr{C}} \frac{\xi^{N}\, e^{\xi l y}\, d\xi}{(\xi-\xi_0)\dots(\xi-\xi_N)}. \tag{36}$$

These may be expanded as sums of exponentials by taking residues at the poles. However, very conveniently, the integral we need in (30) takes the form of a Laplace transform of $S_l(y,0)$, and (36) is the inverse transform. From (36) the transform is

$$C_l = k\int_0^\infty S_l(y,0)\exp(-2ky\cos\beta)\,dy = \frac{k\xi^N}{l(\xi-\xi_0)\dots(\xi-\xi_N)} \tag{37}$$

with $\xi = 2k\cos\beta/l$. This can be expressed as

$$C_l = \{2\cos\beta(1-m\xi_0)\dots(1-m\xi_N)\}^{-1}, \tag{38}$$

where
$$m = l/2k\cos\beta. \tag{39}$$

Since $\xi_{N-j} = \xi_j^*$ and $|\xi_j| = 1$, we have

$$\begin{aligned}(1-m\xi_j)(1-m\xi_{N-j}) &= 1-m(\xi_j+\xi_j^*)+m^2\\ &= (m-\xi_j)(m-\xi_{N-j}),\end{aligned} \tag{40}$$

so the alternative form

$$C_l = \{2\cos\beta(m-\xi_0)\dots(m-\xi_N)\}^{-1}, \quad m = l/2k\cos\beta, \tag{41}$$

can also be used. In particular, when $l = l_0 = \omega^2/g = 4k\sin\beta$ we obtain

$$\gamma(\beta) = C_{l_0} \tag{42}$$

by taking
$$m = m_0 = 2\tan\beta. \tag{43}$$

The values of γ for a range of β are given in table 1. It is found that $\gamma/\cos\beta$ is roughly constant, varying by less than 3 %, so a satisfactory formula for $\tilde{\gamma}$ over this range is

$$\tilde{\gamma} = 0{\cdot}017\omega^3 a_0/g\tan^2\beta. \tag{44}$$

The amplitude at infinity

As $y \to \infty$, the contributions of the poles in (34) decay exponentially, except for the terms from $\xi_0 = i$ and $\xi_N = -i$. These give

$$S_l(y,z) \sim (2D)^{-1} e^{ily+lz} + \text{c.c.}, \quad y \to \infty, \tag{45}$$

where
$$D = (1-w)(1-w^2)\dots(1-w^{N-1}), \quad w = e^{\pi i/N}. \tag{46}$$

β (deg)	$\gamma \times 10^2$	$A(ga_0/\omega^2)^{-\frac{1}{2}}$
45	4·714	7·746
30	5·710	6·833
22·5	6·142	6·523
15	6·483	6·303
11·25	6·607	6·228
9	6·664	6·194
5	6·735	6·153
0	6·767	6·135

TABLE 1. The growth-rate parameter γ and final run-up amplitude A for various beach angles β.

The product DD^* contains all the $2N$th roots of unity except ± 1. Hence

$$DD^* = \lim_{W \to 1} \frac{W^{2N}-1}{(W-1)(W+1)} = N.$$

Moreover, from

$$(1-w^n)/(1-w^{*n}) = -w^n, \quad w^N = -1,$$

we deduce

$$D/D^* = (-1)^{N-1} w^{\frac{1}{2}N(N-1)} = e^{-\frac{1}{2}(N-1)\pi i}.$$

Therefore

$$D = N^{\frac{1}{2}} e^{-\frac{1}{4}(N-1)\pi i}. \tag{47}$$

The asymptotic behaviour (45) represents a wave of constant amplitude on water of infinite depth. With $\phi_0 = a_0 S_l$, the ratio of the amplitude a_∞ at infinity to the amplitude a_0 at the shoreline is

$$a_\infty/a_0 = N^{-\frac{1}{2}} = (2\beta/\pi)^{\frac{1}{2}}. \tag{48}$$

The growth rate $\tilde{\gamma}$ can then be written in terms of a_∞ instead of a_0.

Limiting case $\beta \to 0$, $N \to \infty$

In this limit $m_0 = 2\tan\beta \sim \pi/N$, and from (38) we have asymptotically

$$\log 2C_{l_0} \sim m_0 \sum_{n=0}^{N} \xi_n \sim \frac{i\pi}{N} \sum_{n=0}^{N} w^n$$
$$= -\frac{i\pi}{N}\frac{w+1}{w-1} \to -2,$$

where $w = e^{\pi i/N}$ as before. In the limit $N \to \infty$, therefore,

$$\gamma = C_{l_0} = (2e^2)^{-1} = 0{\cdot}06767. \tag{49}$$

The approach to this value can be seen in table 1.

Similarly one can show in (36) that

$$S_l(y,0) \sim \frac{1}{2\pi i}\int_{\mathscr{C}} \frac{1}{\eta} \exp\left\{\frac{ly}{\beta}\eta - \frac{1}{\eta}\right\} d\eta = J_0\{2(ly/\beta)^{\frac{1}{2}}\} \tag{50}$$

as $\beta \to 0$ for fixed ly/β. This is the result that is given by shallow-water theory. The deep-water wave (45) with non-zero amplitude is lost owing to the non-uniformity of the limit (50) as $ly/\beta \to \infty$, which corresponds to the non-uniform validity of the shallow-water assumptions in the deep water. However, for the integral in (30), this non-uniformity is masked by the additional exponential factor in that integral. If (50) is used with $l/\beta = l_0/\beta = 4k$, we have

$$\gamma = k\int_0^\infty J_0(4(ky)^{\frac{1}{2}})\, e^{-2ky}\, dy = (2e^2)^{-1}. \tag{51}$$

Thus the shallow-water theory gives the correct value for γ in the limit $\beta \to 0$.

6. The solution of the inhomogeneous problem for φ

To describe the further growth of the edge wave and its interaction with the given incident wave, we need the solution of the inhomogeneous problem (25)–(27) for ϕ. The general solution may be written as

$$\phi = a_0 S_{l_0}(y,z) + ib^2(t)\, P(y,z), \tag{52}$$

where $l_0 = \omega^2/g = 4k\sin\beta$ as before, and P is a particular solution satisfying

$$P_{yy} + P_{zz} = 0, \quad -y\tan\beta < z < 0, \tag{53}$$

$$P_y \sin\beta + P_z\cos\beta = 0, \quad -y\tan\beta = z, \tag{54}$$

$$P_z - l_0 P = -\tfrac{1}{2}k^2E^2 = -\tfrac{1}{2}k^2\exp(-2ky\cos\beta), \quad z = 0. \tag{55}$$

The part of the solution for ϕ that represents the *incoming* wave train at infinity must remain unchanged during the growth of the edge wave. Therefore, if $P(y,z)$ is chosen to represent an outgoing wave, the coefficient of S_{l_0} (which then contains all the incoming wave) must remain equal to the undisturbed amplitude a_0. The relation between a_0 and the *incoming* amplitude $\frac{1}{2}a_\infty$ at infinity is given by (48).

We solve the problem for $P(y,z)$ as an expansion in the eigenfunctions $S_l(y,z)$. It is easily shown from Green's theorem that the $S_l(y,0)$ are orthogonal, and the normalization factor may be found from (45) with D given by (47). We have

$$\int_0^\infty S_l(y,0)\, S_m(y,0)\, dy = \frac{\pi}{2N}\,\delta(l-m). \tag{56}$$

We assume that the eigenfunctions $S_l(y,z)$ are complete in order to derive a proposed solution for $P(y,z)$; it may then be verified directly that the proposed solution satisfies all the requirements.

The right-hand side of (55) is expanded as

$$-\frac{kN}{\pi}\int_0^\infty C_l S_l(y,0)\, dl, \tag{57}$$

where

$$C_l = k\int_0^\infty S_l(y,0)\exp(-2ky\cos\beta)\, dy. \tag{58}$$

When $P(y,z)$ is expanded as an integral of the $S_l(y,z)$ over $0 < l < \infty$, we deduce from (55) and (57) that

$$P(y,z) = -\frac{kN}{\pi}\int_0^\infty \frac{C_l}{l-l_0}\, S_l(y,z)\, dl. \tag{59}$$

In (59), the path of integration is indented in $\operatorname{Im} l > 0$ around the pole at $l = l_0$; this ensures that P represents an outgoing wave at infinity. With this choice, it follows from (45) that

$$P(y, 0) \sim ik\frac{p(\beta)}{2D^*}\exp(-il_0 y), \quad y \to \infty, \tag{60}$$

where

$$p(\beta) = 2NC_{l_0}. \tag{61}$$

From (45) and (60), we have

$$\phi e^{i\omega t} \sim \frac{a_0}{2D}\exp(i\omega t + il_0 y) + \frac{a_0 - kpb^2}{2D^*}\exp(i\omega t - il_0 y), \quad y \to \infty. \tag{62}$$

It should be noted that the transform C_l required in the expression for $P(y, z)$ is the same as the one discussed in the last section and given in (41). In the interaction equations, we again are fortunate; only an integral of $P(y, 0)$ is needed and this can be reduced to an integral involving C_l without having to evaluate (59) in further detail.

7. Solution of the interaction equations

The differential equation (24) for b can now be completed with ϕ given by (52). We have

$$\frac{db}{dt} = \frac{\omega^3 \cos\beta}{4g\sin^2\beta}\{\gamma(\beta)\, a_0 b^* - i\mu(\beta)\, kb^2 b^*\}, \tag{63}$$

where

$$\gamma(\beta) = k\int_0^\infty S_{l_0}(y, 0)\, e^{-2ky\cos\beta}\, dy = C_{l_0} \tag{64}$$

as in § 5, and

$$\mu(\beta) = \frac{5}{64\sin 2\beta} - \int_0^\infty P(y, 0)\exp(-2ky\cos\beta)\, dy \tag{65}$$

$$= \frac{5}{64\sin 2\beta} + \frac{N}{\pi}\int_0^\infty \frac{C_l^2}{l - l_0}\, dl. \tag{66}$$

The expression in terms of C_l follows from (59) and a further use of (58).

For small b the term in $b^2 b^*$ is neglected in (63) and we have the initial growth discussed in § 5. When the cubic interaction term is included b grows to the final steady-state value

$$b = b_f = (\gamma a_0 / i\mu k)^{\frac{1}{2}}. \tag{67}$$

If μ is expressed as $-\rho - i\sigma$, and $k = \omega^2/4g\sin\beta$ is introduced, then

$$|b_f|^2 = \frac{4\gamma\sin\beta}{(\rho^2 + \sigma^2)^{\frac{1}{2}}}\frac{ga_0}{\omega^2}, \tag{68}$$

where

$$\rho = -\frac{5}{64\sin 2\beta} - \frac{N}{\pi}\,\mathrm{P}\int_0^\infty \frac{C_l^2}{l - l_0}\, dl, \tag{69}$$

$$\sigma = NC_{l_0}^2 = N\gamma^2, \tag{70}$$

and $\mathrm{P}\int_0^\infty$ denotes the principal value. Here b is the amplitude of the potential χ. Owing to the factor $e^{\frac{1}{2}i\omega t}$ in the complete velocity potential, the surface elevation

β	$\gamma\times 10^2$	$\sigma\times 10^3$	$\rho\times 10^3$	$A(ga_0/\omega^2)^{-\frac{1}{2}}$
45°	4·714	4·444	2·550	7·258
30°	5·710	9·781	9·059	5·853

TABLE 2. Values of the various parameters for $\beta = 45°$ and 30°.

ζ in (4) has amplitude $\frac{1}{2}|b|$. The results are usually expressed in terms of A, the difference between the maximum and minimum run-up on the beach. This is

$$A = |b_f|/\sin\beta = \left\{\frac{4\gamma}{(\rho^2+\sigma^2)^{\frac{1}{2}}\sin\beta}\frac{ga_0}{\omega^2}\right\}^{\frac{1}{2}}. \tag{71}$$

During the growth of the edge wave, the coefficient a_0-kpb^2 of the reflected wave in (62) drops below that of the incident wave, as energy is supplied to the edge wave. In the final steady state it approaches

$$a_r = a_0-kpb_f^2 = (1-p\gamma/i\mu)\,a_0.$$

From (61) and (70), $p\gamma = 2\sigma$. Therefore

$$a_r = \frac{\rho-i\sigma}{\rho+i\sigma}a_0. \tag{72}$$

The reflected amplitude $|a_r|$ returns to a_0 as the edge wave reaches its steady state, but there is a change of phase.

The values of γ for various β have been given in table 1 and those for $\sigma = \pi\gamma^2/2\beta$ follow immediately. From (41), the quantity ρ may be written as

$$\rho = -\frac{5}{64\sin 2\beta}-\frac{1}{8\beta\cos^2\beta}\mathrm{P}\int_0^\infty\frac{dm}{(m-m_0)\,(m-\xi_0)^2\ldots(m-\xi_N)^2}, \tag{73}$$

where $m_0 = 2\tan\beta$ as in (43). There appears to be no simple way of evaluating (73) and, apart from the limiting case $\beta\to 0$, we had to resort to partial fractions. This becomes tedious as N increases. However, we give the results for $\beta = \frac{1}{4}\pi$ and $\frac{1}{6}\pi$ and the asymptotic behaviour for small β. In the next section, a further modifying effect is included and the value of ρ is no longer required to determine the final amplitudes b_f and A.

For $\beta = \frac{1}{4}\pi$ and $\frac{1}{6}\pi$ the results are given in table 2, using (71) for A.

The limiting value for γ as $\beta\to 0$ was found in (49) to be $(2e^2)^{-1}$. Therefore

$$\sigma = N\gamma^2 \sim \pi/8e^4\beta = 7{\cdot}193\times 10^{-3}/\beta. \tag{74}$$

To find the asymptotic behaviour of ρ in (73), it is convenient to use the alternative form obtained from (38) rather than (41), so the factors appear as $\Pi(1-m\xi_n)^2$. In the revised form, after a change of variable to $M = m/m_0$, the integral is

$$\mathrm{P}\int_0^\infty\frac{dM}{(M-1)\,(1-m_0\,\xi_0\,M)^2\ldots(1-m_0\,\xi_N\,M)^2}.$$

As $m_0 = 2\tan\beta \to 0$,

$$\log \prod_{n=0}^{N} (1 - m_0 \xi_n M)^2 \sim -2m_0 M \sum_{0}^{N} \xi_n \sim 2\frac{\pi}{N} Mi \frac{w+1}{w-1} \to 4M.$$

Hence the integral tends to

$$\mathrm{P}\int_0^\infty \frac{e^{-4M}}{M-1}\,dM.$$

A change of variable puts this in the standard form for the Ei function and we have

$$\begin{aligned}\rho &\sim -\left\{\frac{5}{128} + \frac{1}{8}\mathrm{P}\int_0^\infty \frac{e^{-\eta}}{\eta - 4}\,d\eta\right\}\Big/\beta \\ &= 5{\cdot}882 \times 10^{-3}/\beta. \end{aligned} \tag{75}$$

Finally, for the run-up amplitude (71), these results give

$$A \to 5{\cdot}397(ga_0/\omega^2)^{\frac{1}{2}}, \quad \beta \to 0. \tag{76}$$

Again the asymptotic behaviour for ρ checks with the shallow-water result that can be found directly from (50). For then

$$C_l = k\int_0^\infty J_0\left(2\left(\frac{ly}{\beta}\right)^{\frac{1}{2}}\right) e^{-2ky}\,dy = \tfrac{1}{2}e^{-l/2\beta k},$$

and the result follows from (69).

8. The resonance conditions

Guza & Bowen (1976) point out that edge-wave modes which are initially slightly off resonance may grow to a larger final amplitude. We consider the mode with wavenumber k related to ω by

$$k\sin\beta = \omega^2/4g - \tilde{\kappa}a_0, \tag{77}$$

where $\tilde{\kappa}$ is an adjustable parameter. Then there is an extra term $\tilde{\kappa}a_0\chi^{(1)}$ on the right of (18) and this leads to an extra term $-(i\tilde{\kappa}g/\omega)\,a_0 b$ on the right of (24). The amplitude equation (63) becomes

$$\frac{db}{dt} = \frac{\omega^3\cos\beta}{4g\sin^2\beta}\{\gamma a_0 b^* - i\kappa a_0 b - i\mu k b^2 b^*\}, \tag{78}$$

where
$$\kappa = \frac{4g^2\sin^2\beta}{\omega^4\cos\beta}\tilde{\kappa}. \tag{79}$$

For the initial instability,

$$db/dt \propto \gamma a_0 b^* - i\kappa a_0 b, \tag{80}$$

$$d^2b/dt^2 \propto (\gamma^2 - \kappa^2)\,a_0^2 b. \tag{81}$$

The growth rate is greatest for perfect resonance $\kappa = 0$, as expected.

However as the amplitude increases, we have

$$\frac{db}{dt} = \frac{\omega^3\cos\beta}{4g\sin^2\beta}\{(\gamma a_0 - \sigma k b^2)\,b^* - i(\kappa a_0 - \rho k b b^*)\,b\}, \tag{82}$$

where μ has been set equal to $-\rho - i\sigma$ as before. It is easy to show that the final steady amplitude is a maximum when

$$\kappa = \rho k b b^*/a_0, \quad k b^2 = \gamma a_0/\sigma. \tag{83}$$

The second term proportional to b in (82) may be interpreted as a frequency change. As the nonlinear term $\rho b b^*$ becomes important, this frequency change due to the edge-wave self-interactions will detune the *actual* frequency away from resonance. If this is corrected by the adjustment in κ, the system effectively remains on resonance for maximum effect. Thus the final amplitude is greatest for the conditions given in (83). The second term proportional to b^* in (82) may be viewed as the growth term and the steady state is reached when its coefficient is zero as in (83). It may also be noted that the phase difference $\arg b$ is zero in the optimal case, whereas it is not so in the case $\kappa = 0$. This is also related to the phase difference found in (72) for the reflected wave. In the optimal case, $k b_f^2 = \gamma a_0/\sigma$ and $a_r = -a_0$. As the edge wave develops, the coefficient of the reflected wave in (62) changes from a_0 to exactly $-a_0$. This is optimum for energy transfer.

Since $\sigma = N\gamma^2$, the final values for b and A are now

$$b_f = \left(\frac{a_0}{kN\gamma}\right)^{\frac{1}{2}} = \left(\frac{8\beta \sin\beta}{\pi\gamma}\frac{g a_0}{\omega^2}\right)^{\frac{1}{2}}, \tag{84}$$

$$A = \left(\frac{8\beta}{\pi\gamma \sin\beta}\frac{g a_0}{\omega^2}\right)^{\frac{1}{2}}. \tag{85}$$

In this modified case, the integral in ρ is no longer required to determine A, and the results for a full range of β are given in table 1. The limiting value is

$$A \to \left(\frac{16 e^2}{\pi}\frac{g a_0}{\omega}\right)^{\frac{1}{2}} = 6{\cdot}135\left(\frac{g a_0}{\omega^2}\right)^{\frac{1}{2}}, \quad \beta \to 0. \tag{86}$$

A detailed discussion of experiments and a comparison with theoretical results are given by Guza & Inman (1975) and Guza & Bowen (1976). The appearance of edge waves depends strongly on the parameter

$$\epsilon_i = \omega^2 a_0/g \tan^2\beta$$

for the incident wave. For typical values $\beta \approx 4°$–$7°$, $a_0 \approx 2$–6 cm and $2\pi/\omega \approx 2$–4 s, edge waves were observed for ϵ_i ranging roughly from 0·8 to 2·0. The lower limit is associated with amplitudes sufficient to overcome viscous dissipation and to introduce strong enough nonlinear effects. The value $\epsilon_i = 1$ corresponds to the beginning of breaking but there is appreciable reflexion until $\epsilon_i = 2$, which is taken to be the value for the incoming wave to be dissipated in breaking with no reflexion. The production of edge waves by this mechanism seems to be strongly linked to reflective conditions. Although these limits on ϵ_i are not included in the theory, indeed $\epsilon_i \ll 1$ is a formal requirement, the dependence on a_0, ω and β seems to be qualitatively correct for the range in which edge waves are produced. Experimental values of the run-up for a typical case, period $2\pi/\omega = 2{\cdot}7$ s and beach slope $\beta = 6°$, are given in table 3.

Apart from the first entry, which is at the lower limit for the appearance of edge waves, the comparison with (76) or (86) seems reasonable in view of the

a_0 (cm)	1·8	2·0	2·5	3·0	4·0
A (cm)	40	80	100	110	130
$A(ga_0/\omega^2)^{-\frac{1}{2}}$	2·2	4·2	4·7	4·7	4·8

TABLE 3. Experimental estimates of the run-up amplitude A *vs.* incident wave amplitude a_0 for $\beta = 6°$, $2\pi/\omega = 2{\cdot}7$ s.

assumptions that ϵ_i be small and the beach be perfectly reflective in the theoretical analysis.

9. Further study of the perturbation expansions

We now examine the uniform validity of the expansion for the edge-wave solution χ as $y \to \infty$. It is expected, in analogy with travelling edge waves (Whitham 1976), that a non-uniformity in the expansion of χ as $y \to \infty$ leads to a modification in the rate of decay of the edge wave at sea.

To find the appropriate modifications we consider again equation (21) for the determination of $\chi^{(2)}$. Equation (21) is of the same type as the one discussed by Whitham (1976), and the same arguments apply here. The result relevant for this discussion is the asymptotic behaviour of $\chi^{(2)}$ as $y \to \infty$. The first-order contribution comes from the first term in the forcing function for (21) and is given by

$$\chi^{(2)} = -[(i\omega/g)\dot{B} + \tilde{\kappa}B](y\tan\beta + z)E + O(E). \tag{87}$$

Equation (87) shows that $\chi^{(2)}$ is not uniformly $O(\chi^{(1)})$ as $y \to \infty$; therefore the expansion

$$\chi = a_0^{\frac{1}{2}}B\exp(-ky\cos\beta + kz\sin\beta)\cos kx\left\{e^{i\omega t} - a_0\left(\frac{i\omega}{g}\frac{\dot{B}}{B} + \tilde{\kappa}\right)(y\tan\beta + z)\,e^{i\omega t}\right\} + \text{c.c.} \tag{88}$$

is not uniform as $y \to \infty$. However (88) is recognized as the Taylor expansion of the function

$$\chi = a_0^{\frac{1}{2}}B(T)\exp(-ky\cos\beta + kz\sin\beta)\cos kx\exp\left\{i\omega t - a_0\left(\frac{i\omega}{g}\frac{\dot{B}}{B} + \tilde{\kappa}\right)(y\tan\beta + z)\right\} + \text{c.c.}, \tag{89}$$

which is the uniformly valid form. This form can be justified using the method of strained co-ordinates; the modifications are minor but obscure the main steps and will not be repeated here. In the uniform expansion (89) the function $\dot{B}B^{-1}$ gives a dependence of the phase on the offshore co-ordinate, while the term in $\tilde{\kappa}$ leads to a modification in the rate of decay offshore.

When the shallow-water theory is used to calculate the nonlinear correction to the phase the same anomalous behaviour as was found for travelling edge waves (Whitham 1976; Minzoni 1976) is present. The arguments used in that case apply to the present situation, and for a depth distribution of the form $h(y) = \beta y$ for $0 \leqslant y \leqslant l_1$, $h(y) = h_1$ for $y \geqslant l_1$, the change in phase is given by

$$a_0\,\omega g^{-1}\dot{B}B^{-1}\beta(y - l_1)/h_1$$

and the change in exponent by

$$\tilde{\kappa} a_0 \beta (y - l_1)/h_1.$$

These are linear in y, and are in qualitative agreement with the behaviour (89) found using the full nonlinear theory.

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Appendix

The orthogonality condition (13) for the problem

$$\chi_{yy} + \chi_{zz} - k^2\chi = 0, \quad -y\tan\beta < z < 0,$$
$$\chi_y \sin\beta + \chi_z \cos\beta = 0, \quad -y\tan\beta = z,$$
$$\chi_z - k\sin\beta\chi = R, \quad z = 0,$$

is found by applying Green's theorem to χ and the solution

$$E = \exp(-ky\cos\beta + kz\sin\beta)$$

of the homogeneous problem. We have

$$0 = \iint (E\nabla^2\chi - \chi\nabla^2 E)\, dy\, dz$$
$$= \int_B \left(E\frac{\partial\chi}{\partial n} - \chi\frac{\partial E}{\partial n}\right) ds + \int_T (E\chi_z - \chi E_z)\, dy,$$

where B and T indicate the bottom and top surfaces. Since the normal derivatives of both E and χ vanish on the bottom surface, and

$$\chi_z = k\sin\beta\chi + R, \quad E_z = k\sin\beta E$$

on $z = 0$, we have

$$\int_T ER\, dy = 0.$$

This relation is applied to the right-hand side of (21). The contributions of the first term, proportional to B_T, and the third term, proportional to B^2B^*, are immediate; the term B^* requires manipulation. To ease the notation the superscript on $\phi^{(1)}$ will be dropped as in (25)–(27). We need to simplify

$$I = -\int \{\phi_y E_y + \phi_z E_z + \tfrac{1}{2}[(\phi_z - 4k\sin\beta\,\phi)\,E]_z\}\, E dy.$$

First substituting $E_y = -k\cos\beta\, E$ and $E_z = k\sin\beta\, E$, we have

$$I = \int_0^\infty \{k\cos\beta\,\phi_y + \tfrac{1}{2}k\sin\beta\,\phi_z - \tfrac{1}{2}\phi_{zz} + 2k^2\sin^2\beta\,\phi\}\, E^2\, dy.$$

From (25), $\phi_{zz} = -\phi_{yy}$. Integration by parts is then used to eliminate y derivatives of ϕ and we have

$$I = -[\tfrac{1}{2}\phi_y + 2k\cos\beta\phi]_{y=0} + (4k^2\cos^2\beta + 2k^2\sin^2\beta)\int_0^\infty \phi E^2\,dy + \tfrac{1}{2}k\sin\beta\int_0^\infty \phi_z E^2\,dy.$$

From the boundary condition (27) with $\omega^2/g = 4k\sin\beta$,

$$\phi_z = 4k\sin\beta\phi - \tfrac{1}{2}ik^2E^2B^2;$$

therefore

$$I = -[\tfrac{1}{2}\phi_y + 2k\cos\beta\phi]_{y=0} + 4k^2\int_0^\infty \phi E^2\,dy - \tfrac{1}{4}ik^3\sin\beta\int_0^\infty E^4\,dy\,B^2.$$

Using both (26) and (27) at the origin, we have

$$\phi_y = -\phi_z\cot\beta = -4k\cos\beta\phi + \tfrac{1}{2}ik^2B^2\cot\beta$$

there. Finally, then,

$$\begin{aligned} I &= -\tfrac{1}{4}ik^2B^2\cot\beta + 4k^2\int_0^\infty \phi E^2\,dy - \tfrac{1}{4}ik^3\sin\beta\int_0^\infty E^4\,dy\,B^2 \\ &= 4k^2\int_0^\infty \phi E^2\,dy - \frac{3ik^2B^2}{8\sin 2\beta}. \end{aligned}$$

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